ETH zürich

Institute for Dynamic Systems and Control

Control Systems 1

Johannes Schulte-Vels

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Welcome!

Polybox

PW: jschul

Website

jschultev.github.io/personal_website

Today

- Repetition Session 6 plus:
  - Effect of Poles and Zeros
  - Pole - Zero Cancellation
- Theory Recap
  - General Idea
  - Root Locus
- Q&A Session / Done

Repetition Session 6

Transfer Function Notations

$\underbrace{\text{Transfer Function}}_{\text{Let's call it «TF»}}$

Let's call it «TF»

On the right we have the general form, where we have the ratio of two polynomials + «feedthrough» d:

$$\boxed{G(s) = \frac{N(s)}{D(s)} = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + d.}$$

There are however some more notations, that can come in quite handy, depending on our goal:

Partial Fraction Expansion

$$\boxed{G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n} + d,}$$

Root Locus Form (next week)

$$\boxed{G(s) = \frac{k_{rl}}{s^q}\,\frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_{n-q})},}$$

Bode Form (in 3 weeks)

$$\boxed{G(s) = \frac{k_{\text{Bode}}}{s^q}\,\frac{\left(\frac{s}{-z_1} + 1\right)\left(\frac{s}{-z_2} + 1\right)\cdots\left(\frac{s}{-z_m} + 1\right)}{\left(\frac{s}{-p_1} + 1\right)\left(\frac{s}{-p_2} + 1\right)\cdots\left(\frac{s}{-p_{n-q}} + 1\right)}}$$

Transfer Functions Important Note

$$G(s) = \boxed{\frac{N(s)}{D(s)}} = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + d. \quad = \frac{k_{rl}}{s^q} \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_{n-q})},$$

- Zeros = Roots of Numerator
- Poles = Roots of Denominator

See how in the Root Locus we have all the roots of N and D in **factorized** notation.

Generally for LTI SISO systems, de degree of the **Denominator** is **higher** than the **Numerator**.
Also, we often times say **d = 0** for simplicity.

Transfer Function Visualisation

$G(s) = M \cdot e^{j\varphi}$ / φ = phase

magnitude

Remember how the TF is just a complex number if a certain input is given?
Lets use this to see how the transfer function changes a certain input (similar example as before):

$$\boxed{u(t) = \sin(t) \qquad y_{ss}(t) = |G(j)| \sin(t + \angle G(j))} \leftarrow s = j$$

So how do we actually determine the magnitude and phase of the transfer function?
Consider the factorized TF:

$$\boxed{G(s) = 2\frac{s+1}{(s+2)(s+1+j)(s+1-j)}}$$

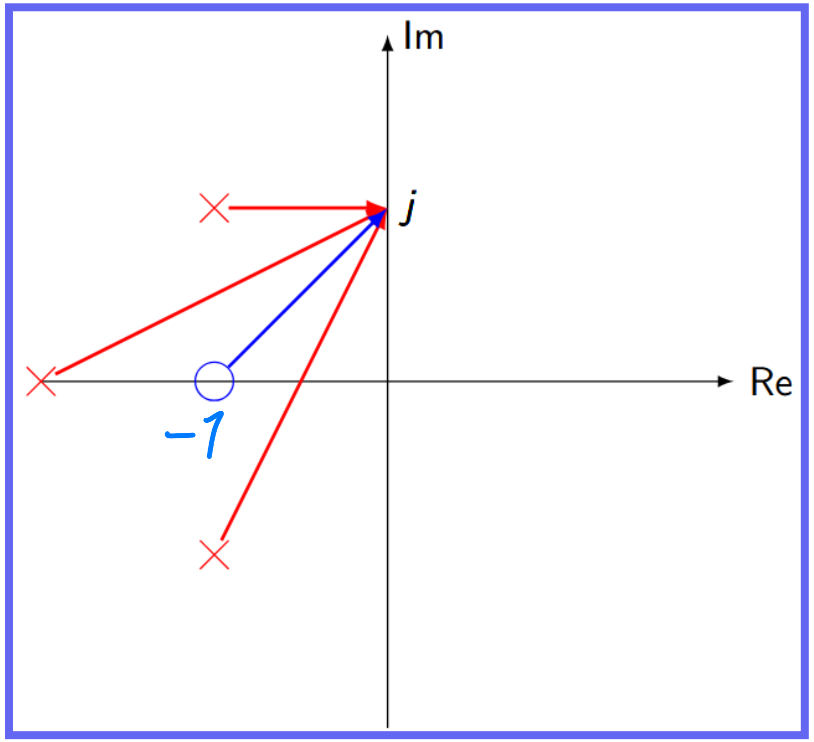
We will look at two ways on how to determine magnitude an phase: **Computationally and Graphically**

Magnitude

$s = j$

Generally for magnitudes: $|a \cdot b| = |a| \cdot |b|$

Our TF becomes: $|G(s)| = 2\frac{|s+1|}{|s+2| \cdot |s+1+j| \cdot |s+1-j|}$



Graphically speaking, |s - p| is just the length of the vector from p to s

Reading from the plot (knowing the position of poles and zeros)

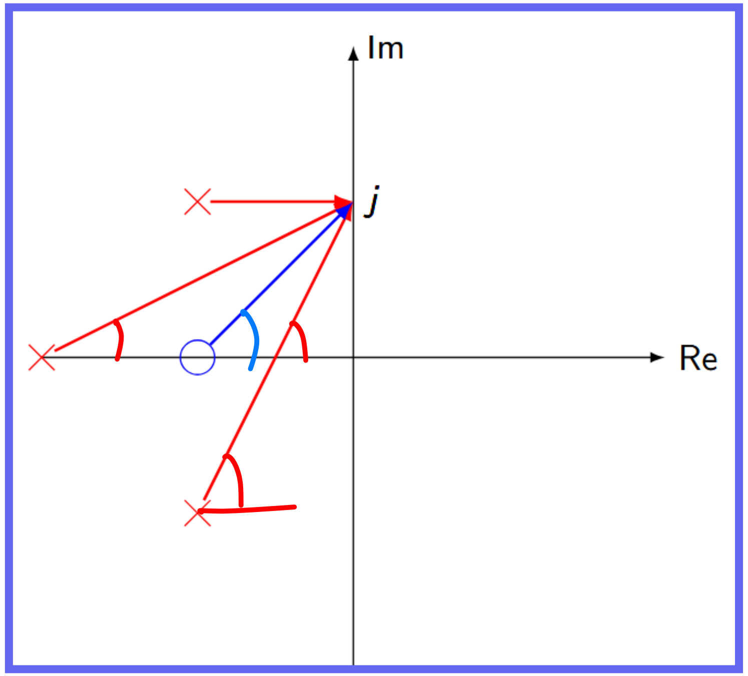
$$|G(j)| = 2\frac{\sqrt{2}}{\sqrt{5} \cdot \sqrt{5} \cdot 1} = \frac{2\sqrt{2}}{5}$$

Phase

Generally for magnitudes: $\angle(a \cdot b) = (\angle a) + (\angle b)$

Our TF becomes:

$$\angle G(s) = \angle(2) + \angle(s+1) - \angle(s+2) - \angle(s+1+j) - \angle(s+1-j)$$



Graphically speaking $\angle(s-p)$ is just the angle formed by vector from p to s with the real axis

Reading from the plot (lowkey):

$$\angle G(j) = 0 + 45° - \arctan(1/2) - \arctan(2) - 0° = -45°$$

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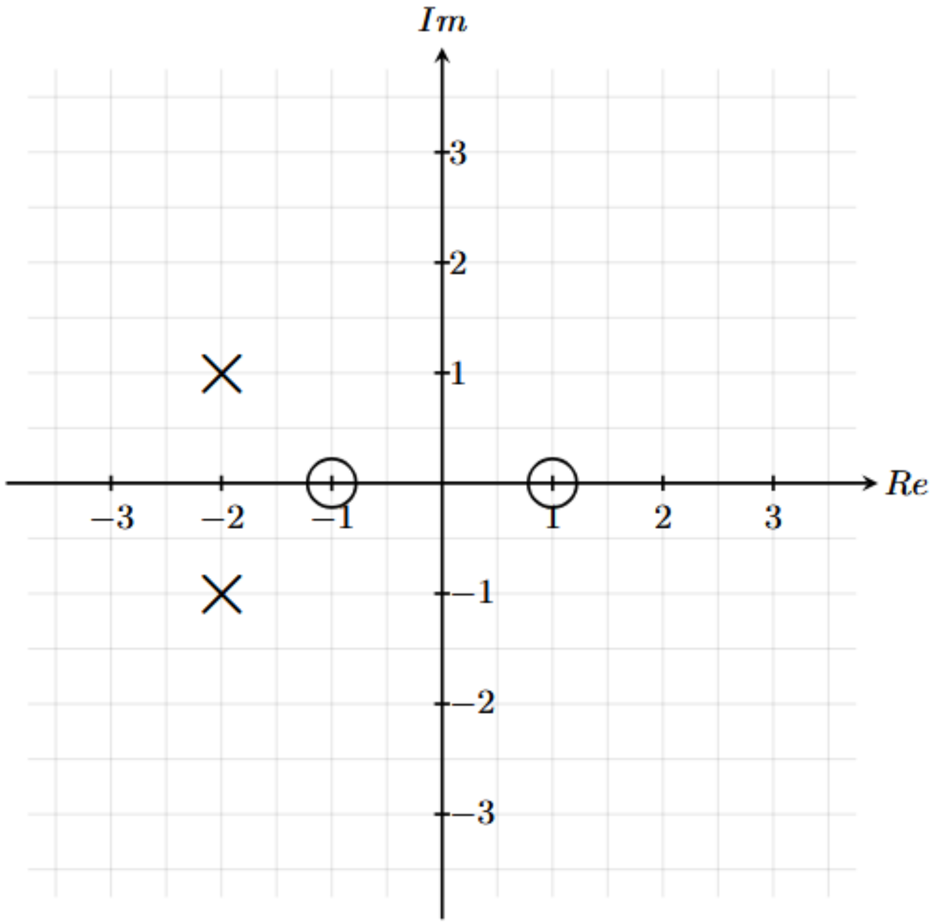

What is the phase $\angle G(j)$ of $G(s = j)$?

A) $= 45°$

B) $= 0°$

C) $= -45°$

D) $= 135°$

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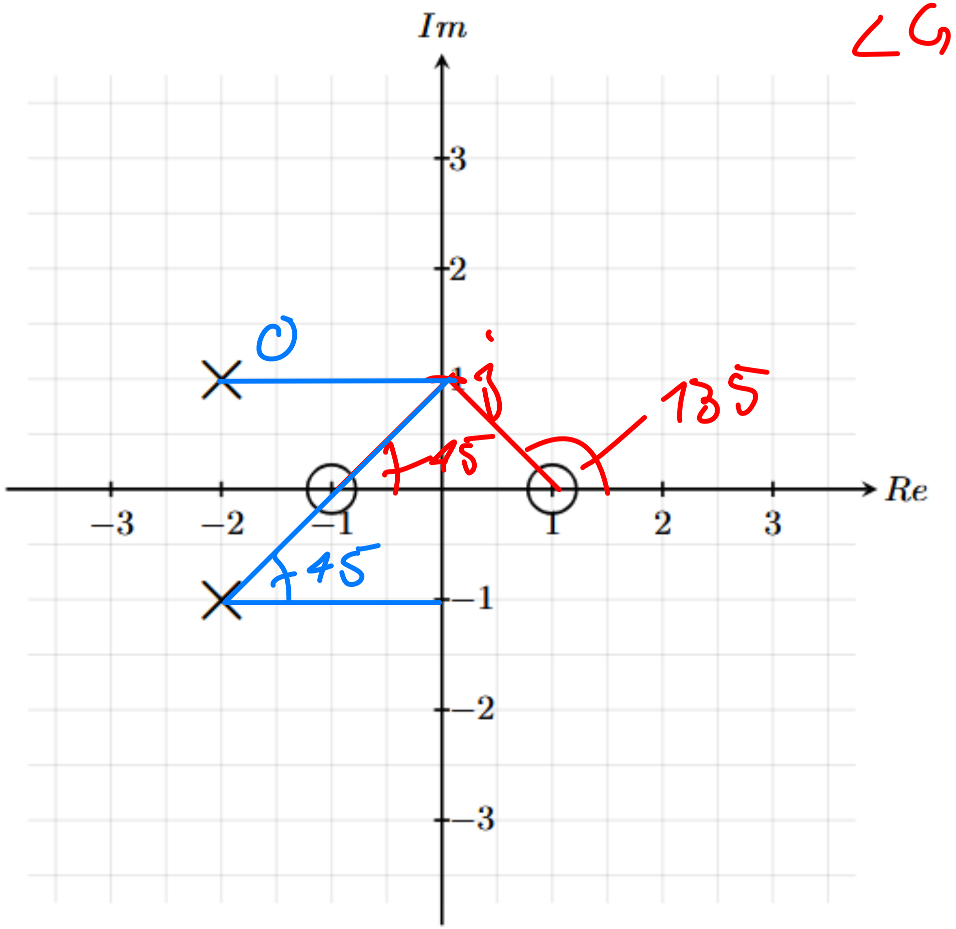
$$G(s) = \frac{(s-1)\cdot(s+1)}{(s+2-j)\cdot(s+2+j)}$$

$$\angle G(j) = 135 + 45 - 0 - 45 = \underline{\underline{135}}$$



What is the phase $\angle G(j)$ of $G(s = j)$?

A) $= 45°$

B) $= 0°$

C) $= -45°$

D) $= 135°$

Effect of Poles

Effect of Poles

$f(t)$	$\mathscr{L}f(t) = F(s)$	$t^n \ (n = 0, 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$
1	$\frac{1}{s}$	$\sin kt$	$\frac{k}{s^2+k^2}$
$e^{at}f(t)$	$F(s-a)$	$\cos kt$	$\frac{s}{s^2+k^2}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$	$\sin^2 kt$	$\frac{2k^2}{s(s^2+4k^2)}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	$\cos^2 kt$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
$\delta(t)$	1	e^{at}	$\frac{1}{s-a}$
$\delta(t-a)$	e^{-sa}		
$\frac{d^n}{dt^n}\delta(t)$	s^n		

Let us consider how the system reacts to a dirac impulse input with zero initial condition.

$$x(0) = 0 \quad u(t) = \delta(t)$$

Looking at the s - domain answer and transforming the input, we get:

$Y(s) = G(s) \cdot U(s)$

$$U(s) = 1. \quad \rightarrow \quad Y(s) = G(s) \cdot 1 = G(s)$$

Now we can already go back to the t - domain:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)\}$$

Effect of Poles

Now lets have a look at a transfer function in partial fraction expansion:

$$G(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n}$$

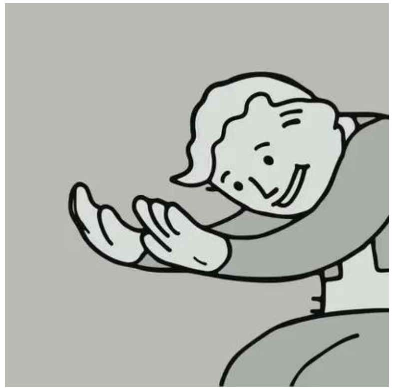
Luckily, in this form it is super easy to take the inverse laplace transform

$\cdot r_i$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{r_i}{s-p_i}\right\} = r_i e^{p_i t}}$$

And now since the laplace transform is a linear transformation the output just becomes:

$$\boxed{y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_n e^{p_n t}, \quad t \geq 0}$$



Hold up, this looks familiar…?

Effect of Poles

Let’s try to find this solution differently by applying the same input in the t - domain.

$$y(t) = C\cancelto{0}{e^{At}}x_0 + C\int_0^t e^{A(t-\tau)}\,B\,u(\tau)\,\mathrm{d}\tau + D\cancelto{0}{u}(t). \qquad \text{with} \qquad D = 0,\; x(0) = 0,\; u(t) = \delta(t)$$

$$y_{imp}(t) = C\int_0^t e^{A(t-\tau)}\,B\,\delta(\tau)\,\mathrm{d}\tau = \boxed{Ce^{At}B}$$

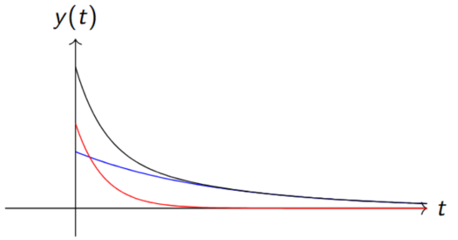
Remember how for the initial condition response, every eigenvalue of the matrix A was found in an exponential term and helped us to construct a solution?

Here, our time response is given by this exact form and matches with the result we got via laplace transform!

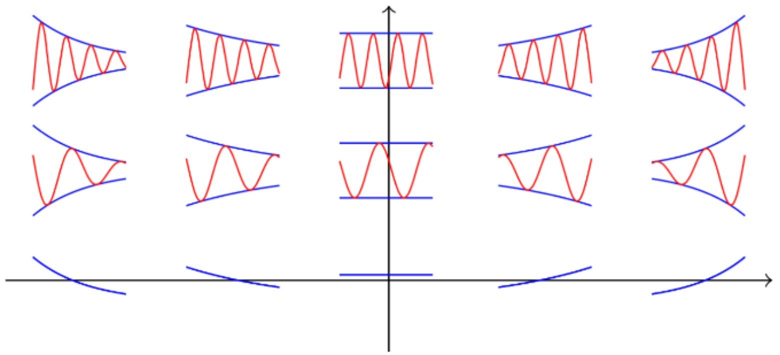
Why though??

Example

$$y(t) = Ce^{At}x_0 = \begin{bmatrix} c_1 & c_2 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{bmatrix}\begin{bmatrix} x_{0,1} \\ x_{0,2}\end{bmatrix} = c_1 e^{\lambda_1 t}x_{0,1} + c_2 e^{\lambda_2 t}x_{0,2}$$



Effect of Poles



Remember the derivation for the TF?

$$G(s) = C(sI - A)^{-1}B + D = \frac{N(s)}{D(s)}$$

Now considering that for a general matrix $\boldsymbol{A}$, $A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$

$$\boxed{G(s) = \frac{1}{\det(sI - A)} \underbrace{C\ \mathrm{adj}(sI - A)\ B + D}_{N(s)} = \frac{N(s)}{\det(sI - A)}}$$

So now wonder that the roots of the Denominator in the TF are the same as the eigenvalues of matrix A!

They are therefore also important for stability assessment!


Laplace?
No Laplace?

Effect of Zeros

Effect of Zeros

Lets start by considering a system where the following input - output relation is given:

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(t)$$

As usual we apply our fundamental input $u(t) = e^{st}$ which will lead us to:

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} e^{st} = s e^{st}$$

$s = G(s)$

$Y(s) = G(s)\, U(s)$

So, this tells us, that a differentiation relation is given by the TF:

$$\boxed{G_{\text{diff}}(s) = s}$$

Lets keep this in mind and consider an example!

Effect of Zeros

$-\frac{1}{z}\cdot(s - z)$

Consider the TF $\tilde{G}(s)$ with an added zero:

$$G(s) = \left(\frac{s}{-z} + 1\right)\tilde{G}(s) = \tilde{G}(s) + \frac{s}{-z}\tilde{G}(s)$$

$\tilde{G}(s) = \frac{1}{s+1}$

$G(s) = \frac{s-3}{s+1}$

When now going back to the time - domain response, remember the s means taking a derivative. Therefore:

derivative

$$\boxed{y(t) = \tilde{y}(t) + \frac{1}{-z}\dot{\tilde{y}}(t)}$$

This means that we usually have an **anticipatory** effect. However, we will discuss to cases:

Effect of Zeros

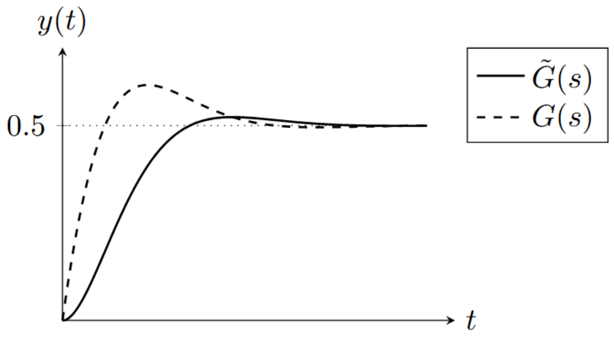
z= position of zero

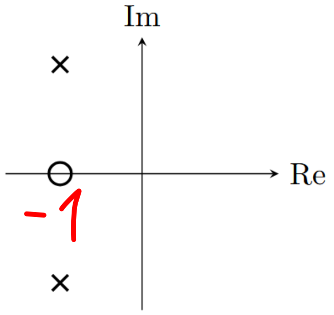
$$\boxed{y(t) = \tilde{y}(t) + \frac{1}{-z}\dot{\tilde{y}}(t)}$$

Minimum Phase Zero

When the real part of the zero lies in the **left half plane**, it is called minimum phase. This adds a **positive** derivative action to the output.

$y(t) = \tilde{y}(t) + \dot{\tilde{y}}(t)$

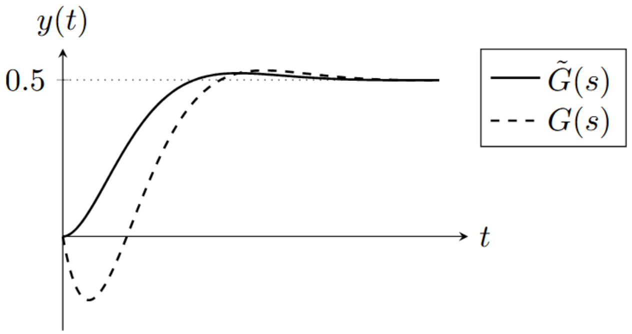


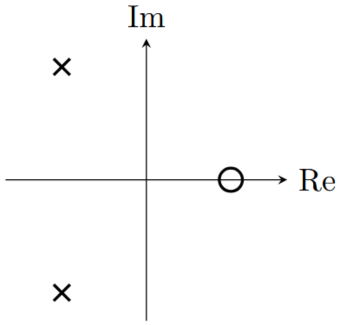


Non Minimum Phase Zero

When the real part of the zero lies in the **right half plane**, it is called non minimum phase. This adds a **negative** derivative action to the output (not good for controls)

$y = \tilde{y} - \dot{\tilde{y}}$





Pole - Zero Canellation

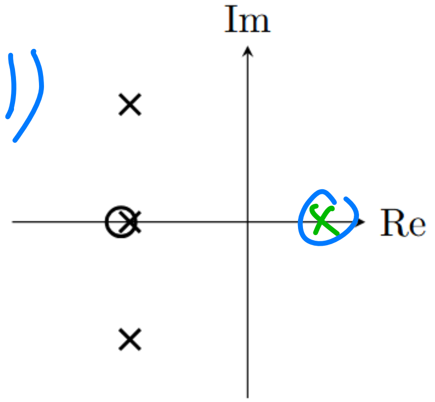
Pole - Zero Cancellation

We would like to analyse what happens when a pole and a zero are close to each other, and or if we would cancel them in the TF. Consider the following TF;

$(s-(-1-\varepsilon))$

$\cancel{(s+1)}$

$$G(s) = \frac{s+1+\varepsilon}{\cancel{(s+1)}(s+1+j)(s+1-j)}$$



Remember that looking at the partial fraction expansion, we actually see how much a pole is present in the response. So let's compute the residues:

$\varepsilon \cdot e^{-1t}$

$$r_1 = \lim_{s\to-1}(s+1)G(-1) = \varepsilon$$

$$r_2 = \lim_{s\to-1-j}(s+1+j)G(-1-j) = \frac{j-\varepsilon}{2}$$

$$r_3 = \lim_{s\to-1+j}(s+1-j)G(-1+j) = \frac{-j-\varepsilon}{2}$$

$$\boxed{G(s) = \frac{\varepsilon}{s+1} + \frac{0.5(j-\varepsilon)}{s+1+j} - \frac{0.5(j+\varepsilon)}{s+1-j}}$$

Pole - Zero Cancellation

$$\boxed{G(s) = \cancelto{0}{\frac{\varepsilon}{s+1}} + \frac{0.5(j-\varepsilon)}{s+1+j} - \frac{0.5(j+\varepsilon)}{s+1-j}}$$

Remember that we chose the zero being arbitrarily close to the pole. We can see now that when we **move the zero closer to the pole**, the residue of this pole**, it’s effect on the response, gets smaller**, until it gets 0 eventually.

This is fine as long as we do not cancel any unstable poles. **DO NOT CANCEL UNSTABLE POLES!** (System blows up).

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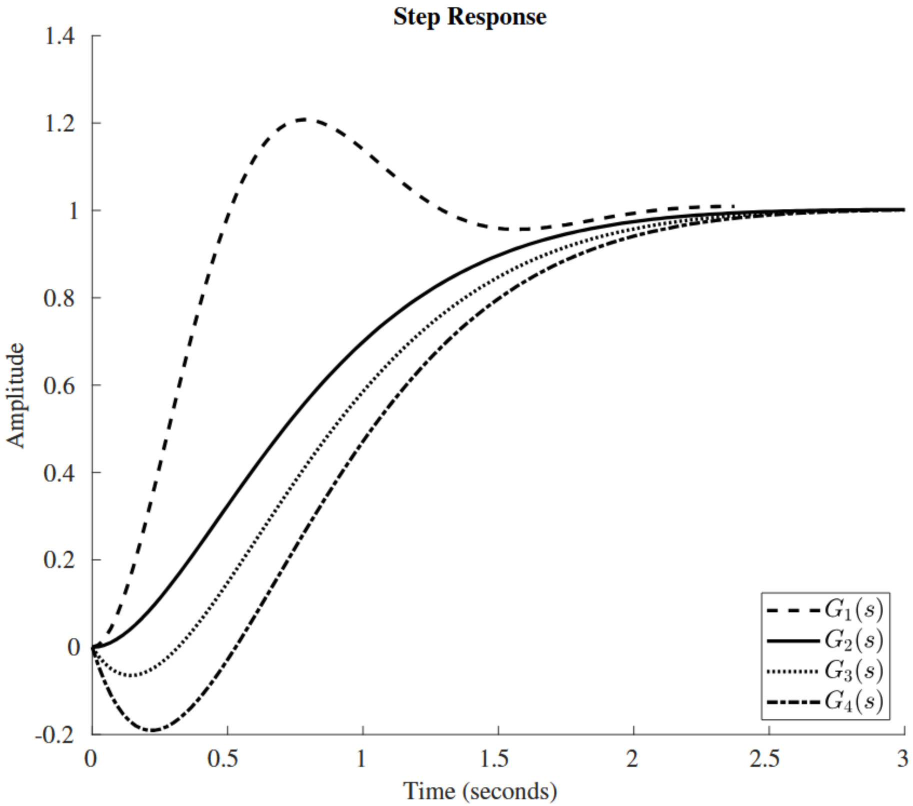

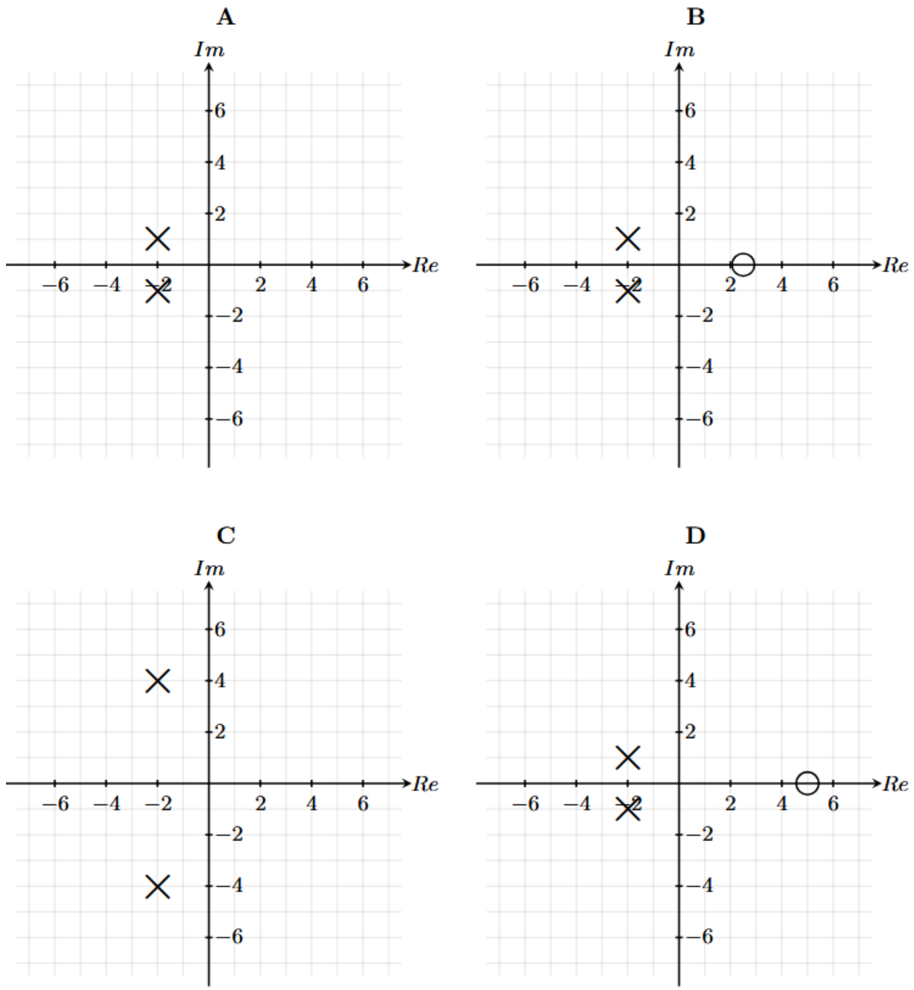

A) (A, G_1), (B, G_4), (C, G_2), (D, G_3)

C) (A, G_3), (B, G_2), (C, G_4), (D, G_1)

B) (A, G_2), (B, G_3), (C, G_1), (D, G_4)

D) (A, G_2), (B, G_4), (C, G_1), (D, G_3)

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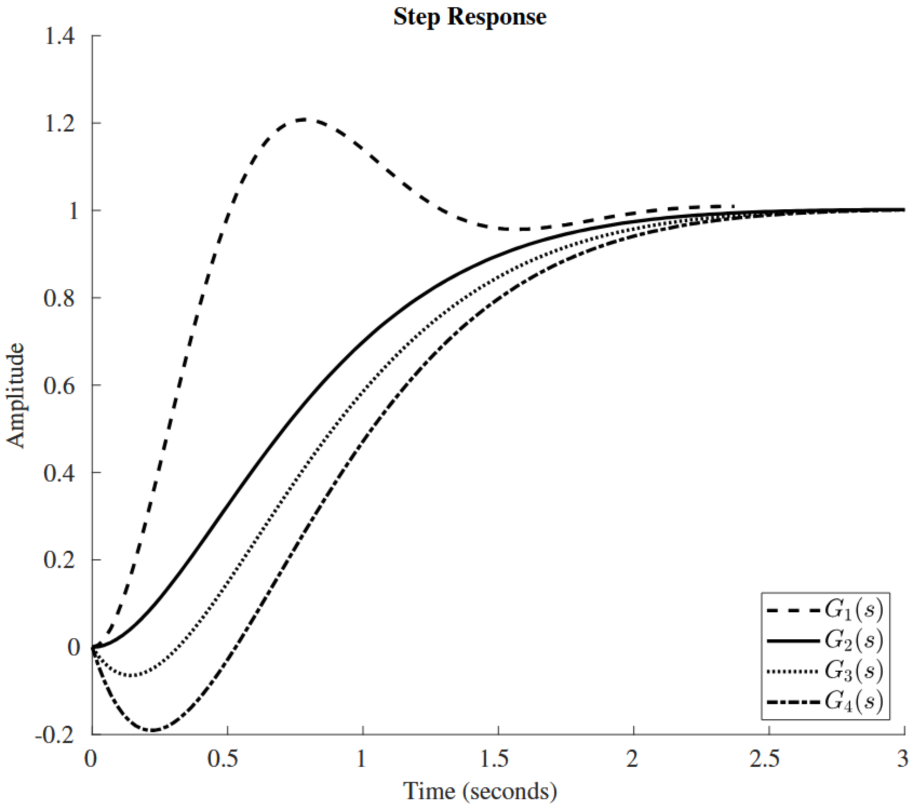

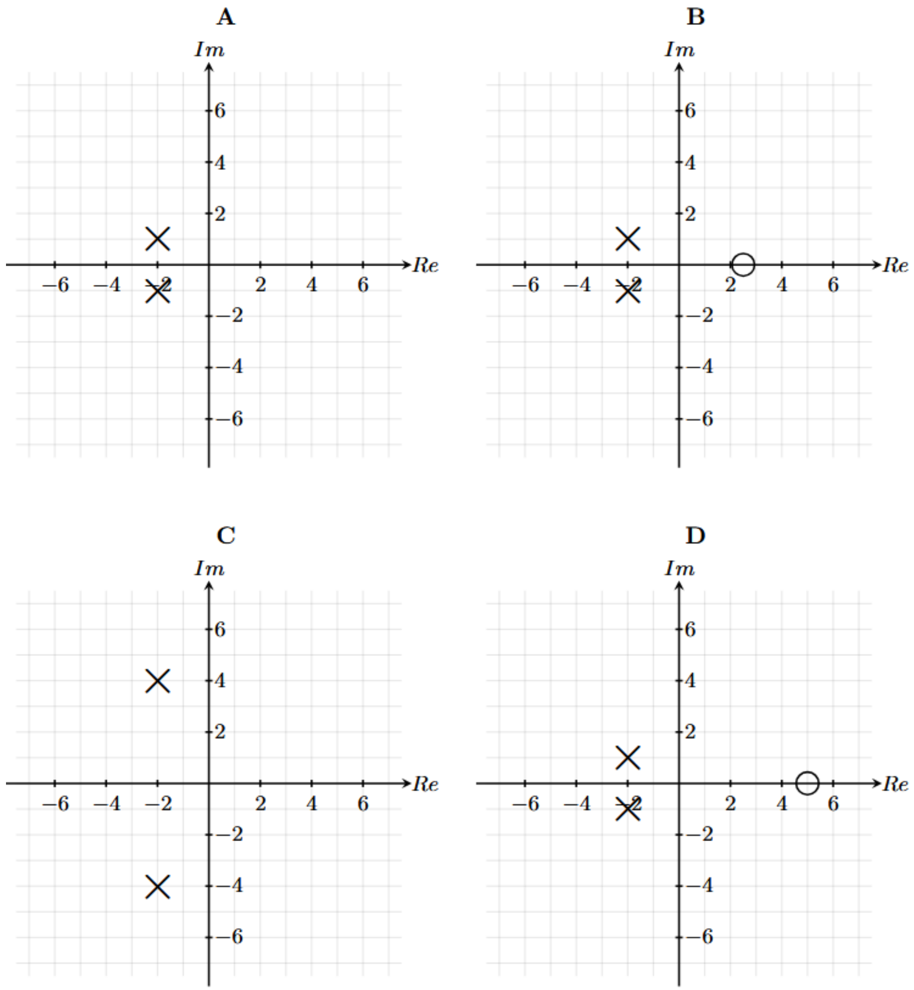

A) $(A, G_1), (B, G_4), (C, G_2), (D, G_3)$

C) ~~$(A, G_3), (B, G_2), (C, G_4), (D, G_1)$~~

B) ~~$(A, G_2), (B, G_3), (C, G_1), (D, G_4)$~~

D) $(A, G_2), (B, G_4), (C, G_1), (D, G_3)$

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Q22 **(1 Points)** Mark all correct statements.

- [A] In order to assess the contribution of different poles to the time response of a linear time-invariant system G it is useful to look at the transfer function of G in partial fraction expansion.
- [B] The step response of a linear time-invariant system is the same as its initial condition response with initial condition $x(0) = B$.
- [C] Performing a pole-zero cancellation in the transfer function of a linear time-invariant system does not pose threats in terms of misleading (internal) stability assessments, if and only if, the zero that is cancelled is a minimum phase zero.
- [D] For linear time-invariant systems, zeros can be interpreted as adding derivative action to the output with anticipatory effect.

A | C

B | D

HS 2023

Q22 **(1 Points)** Mark all correct statements.

- A In order to assess the contribution of different poles to the time response of a linear time-invariant system G it is useful to look at the transfer function of G in partial fraction expansion.
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- C Performing a pole-zero cancellation in the transfer function of a linear time-invariant system does not pose threats in terms of misleading (internal) stability assessments, if and only if, the zero that is cancelled is a minimum phase zero.
- D For linear time-invariant systems, zeros can be interpreted as adding derivative action to the output with anticipatory effect.

A	C
B	D

Theory Recap

General Idea Now

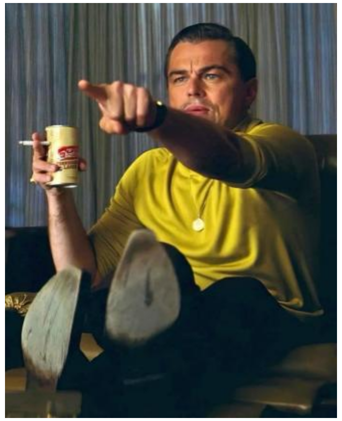
Remember Our Course Objectives?

1. **Modeling:**
   – Learn how to represent a dynamic system in such a way that it can be treated effectively using mathematical tools. ← State Space Representation

2. **Analysis:**
   – Understand the basic characteristics of a system (e.g., stability), and how the input affects the output. ← Stability, Poles, Zeros, Transfer Function

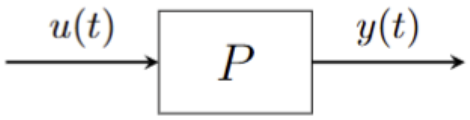
3. **Synthesis:**
   – Figure out how to change a system, typically by feedback, in such a way that it behaves in a desirable way. ← **From now on!**

Synthesis

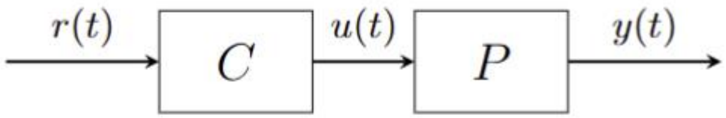


So we want to make the system do what we want. We want to **control** (the) **system!**

To achieve this, we must design controlers (mechanisms) that actively influence the system. At first we can think of a controller just as a mechanism converting a reference signal **r** to suiting input **u**



(a) Open-loop system.

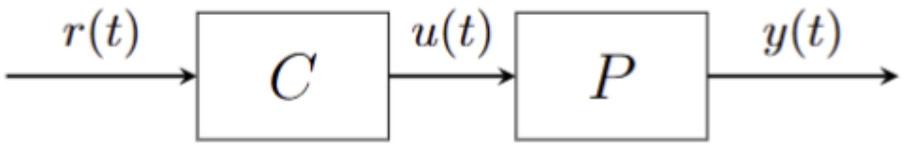


(b) Open-loop system with controller.

But for it to get interesting, let us introduce feedback!

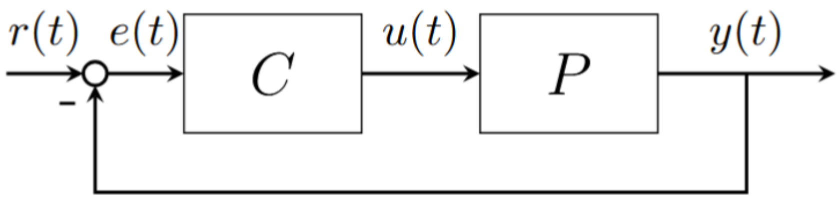
Closed Loop System

Open-loop system



- No feedback ➞ Input doesn't depend on output
- Simple but unprecise

Closed-loop system
Feedback

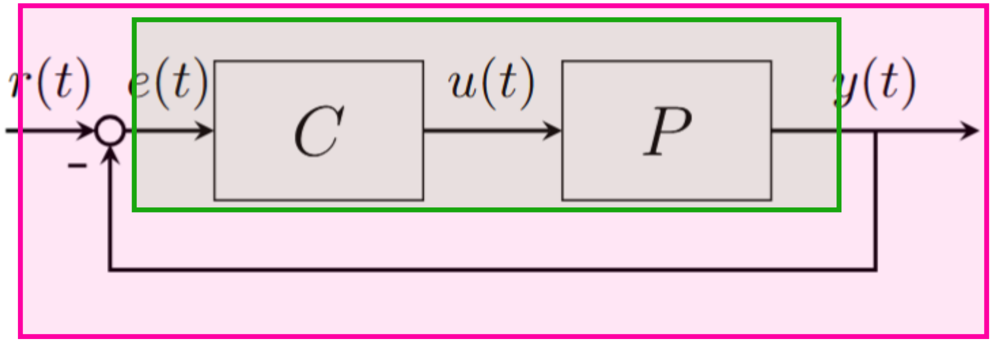


- Feedback! ➞ Input depends on output
- More complex
- Can become unstable (we will later look at what that means)

Over the next weeks we are going to build intuition (also mathematically) what feedback really does and how we can use it to control our system

Feedback Control

Different Transfer Functions



Let us consider the general Closed Loop Feedback System.
Now we want to define certain signal mappings (with more to follow):

open-loop (O-L) TF $(e \rightarrow y)$:

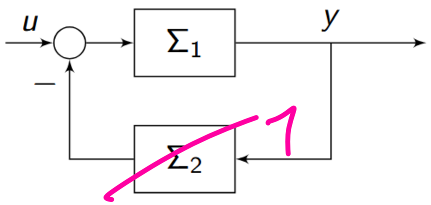
$$\boxed{L(s) = C(s)P(s)}$$

closed-loop (C-L) TF $(r \rightarrow y)$:

$$\boxed{T(s) = \frac{L(s)}{1+L(s)} = \frac{C(s)P(s)}{1+C(s)P(s)}}$$

(aka. Complementary sensitivity function)

Throwback



$$\Sigma = \frac{\Sigma_1}{(1+\Sigma_1\Sigma_2)}$$

$$\Sigma = \frac{\Sigma_1}{1+\Sigma_1}$$

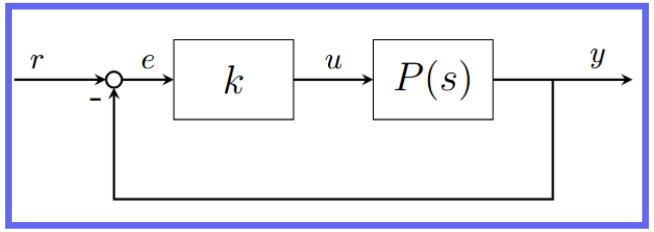
$$\Sigma_1 = L(s)$$

Root Locus Method

Motivation

We call this setup **proportional control**, since the controler only multiplies our system with a scalar gain

Let us consider the following system



The controller C is given by: $\boxed{k}$

The plant P is given by: $\boxed{P(s) = \frac{1}{s-1}}$

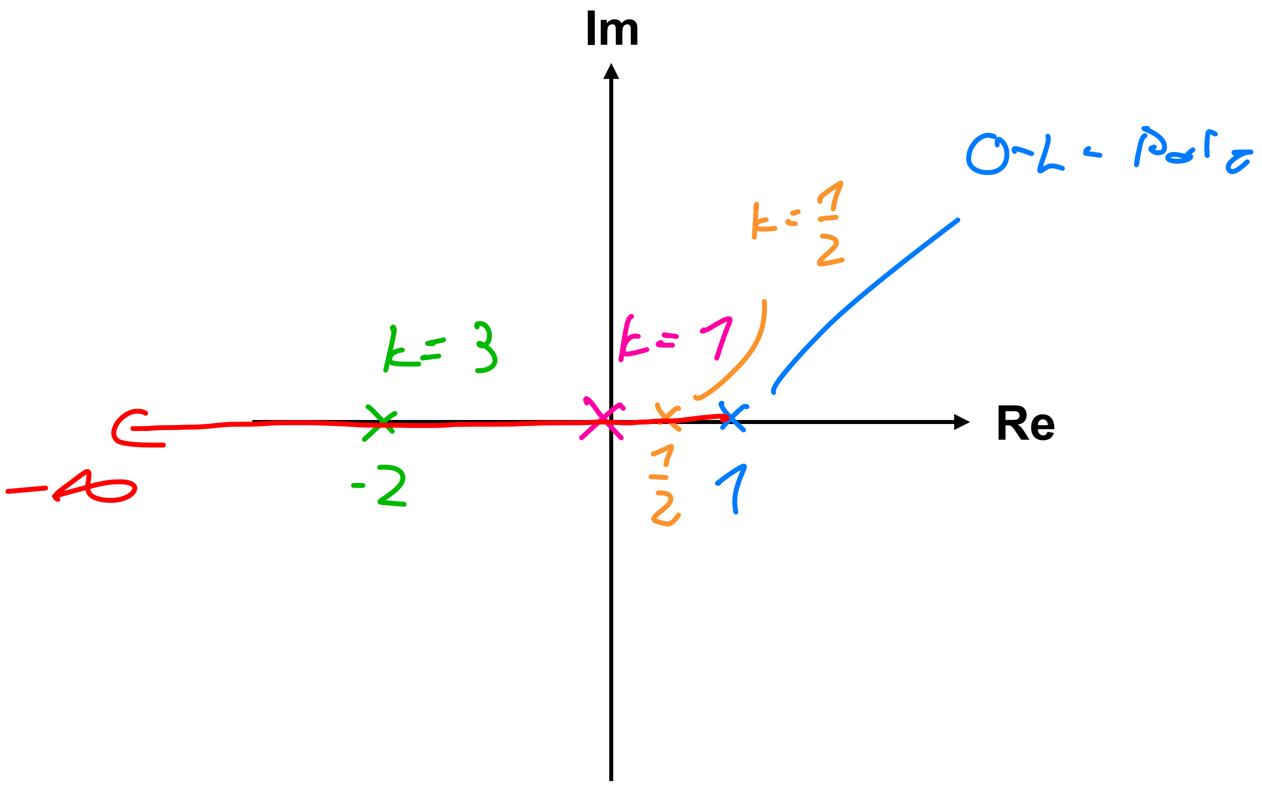
Now the O-L TF: $\boxed{L(s) = kP(s) = k\frac{1}{s-1}}$

Now the ~~O-L~~ C-L TF: $\boxed{T(s) = \frac{kP(s)}{1+kP(s)} = \frac{k\frac{1}{s-1}}{1+k\frac{1}{s-1}} = \frac{k}{s+k-1}}$

Lets plot the poles and zeros of our TF, O-L and C-L!

Motivation

$$L(s) = \frac{k}{s-1} \qquad T(s) = \frac{k}{s+k-1} \approx \lim_{k\to\infty} \frac{k}{s+k}$$



Motivation 2

Because it's been so great, let's look at another example!!

$$\boxed{P(s) = \frac{1}{s(s+4)}}$$

$$\boxed{T(s) = \frac{k}{s^2+4s+k},}$$

We find the C-L poles (by setting denominator to zero):

$$\boxed{p_1 = -2 + \sqrt{4-k}, \quad p_2 = -2 - \sqrt{4-k}.}$$

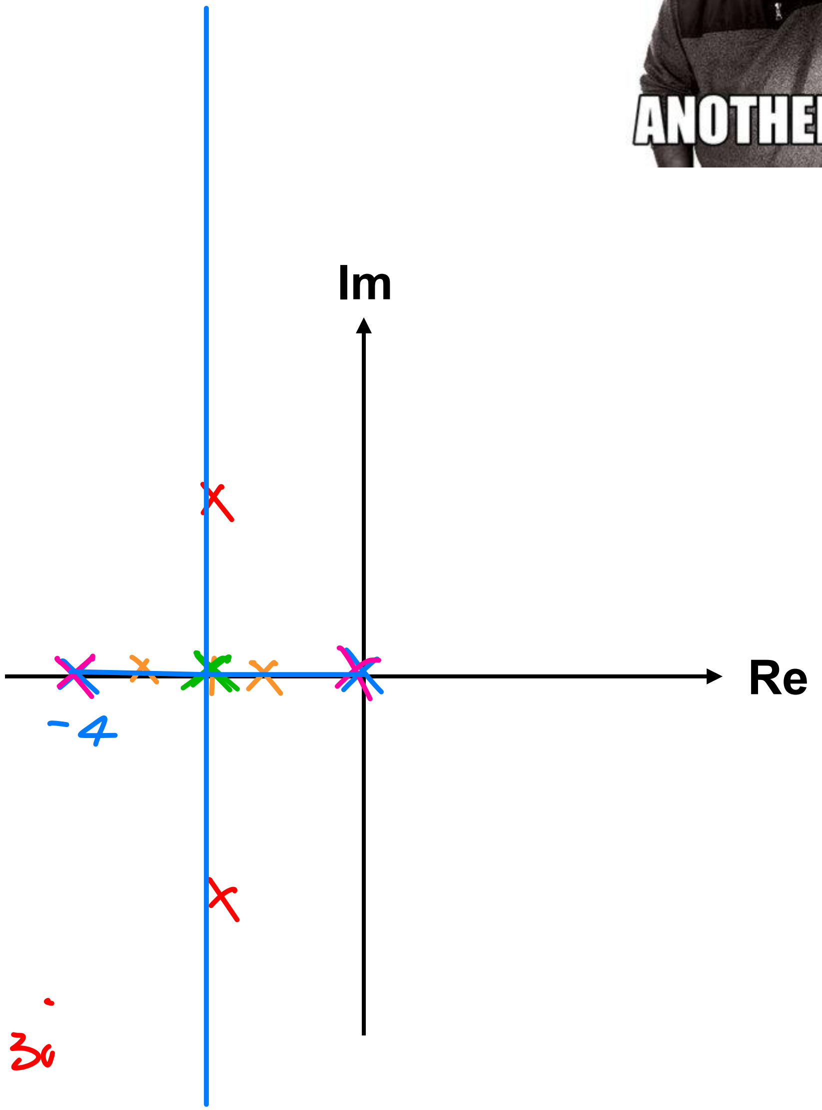
$p_1 = -2+2$, $p_2 = -2-2$

$= 0$ $\qquad = -4$

$k=2$ $\quad = -2+\sqrt{2}$ $\quad = -2-\sqrt{2}$

$k=4$ $\quad = -2$ $\quad = 2$

$k=13$ $\quad = -2+3i$, $\quad = -2-3i$



Motivation Root Locus

Okay, so now we have seen how different k values influence the C-L poles.

But wouldn't it be nice to see a continuous map of all the poles (and also zeros) of the closed-loop TF? To see where our poles end up when increasing (or decreasing) k.

For this we will introduce the **Root Locus.** It is basically what we did before, but it will make our lifes easier by introducing general rules, so that **we do not have to calculate every position** for every k.

Root Locus means nothing but «The location (locus) of the roots» and will basically be a plot.

BTW: Later we can also introduce more complex kinds of controlling, not only proportional control, and see the advantages of the root locus

Root Locus Rules

There are quite some rules for the root locus plot. If you do not trust them, make all the calculations and see that you will end up with the same result.
I will try giving you some intuition for it.

Put them on your Cheat Sheet! (Eventually, you'll know them by heart though

Root Locus Rules

1. Root Locus is always **symmetric** with respect **to the real axis**
2. Root Locus always has **as many lines as # of O-L poles** (same degree)
3. Root Locus **starts at O-L poles and ends at O-L zeros** (exception: **Rule 5**)
4. All points on the real axis lie on the root locus. **(sketch from right to left!!!)**
   All points to the left of an **odd** number of zeros + poles are on the **positive** root locus
   All Point to the left of an **even** number of zeros + poles are on the **negative** root locus
5. If there are not enough zeros for every pole, the poles go to infinity along asymptotes

Center of Asympt = $\boxed{S_{COM} = \frac{\sum x_p - \sum x_z}{n-m}}$

n = #Poles
m = #Zeros

$\boxed{\angle(s) = \frac{(2q+1)\cdot 180^\circ}{n-m} \quad q = \{0, 1, ..., n-m-1\}}$ = Angle of Asympt.

Some extra rule:

1. Lines enter and leave real axis at 90 degrees

Examples

Root Locus Rules

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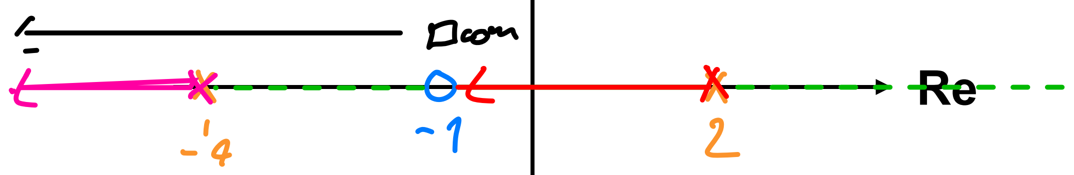
Some extra rule:

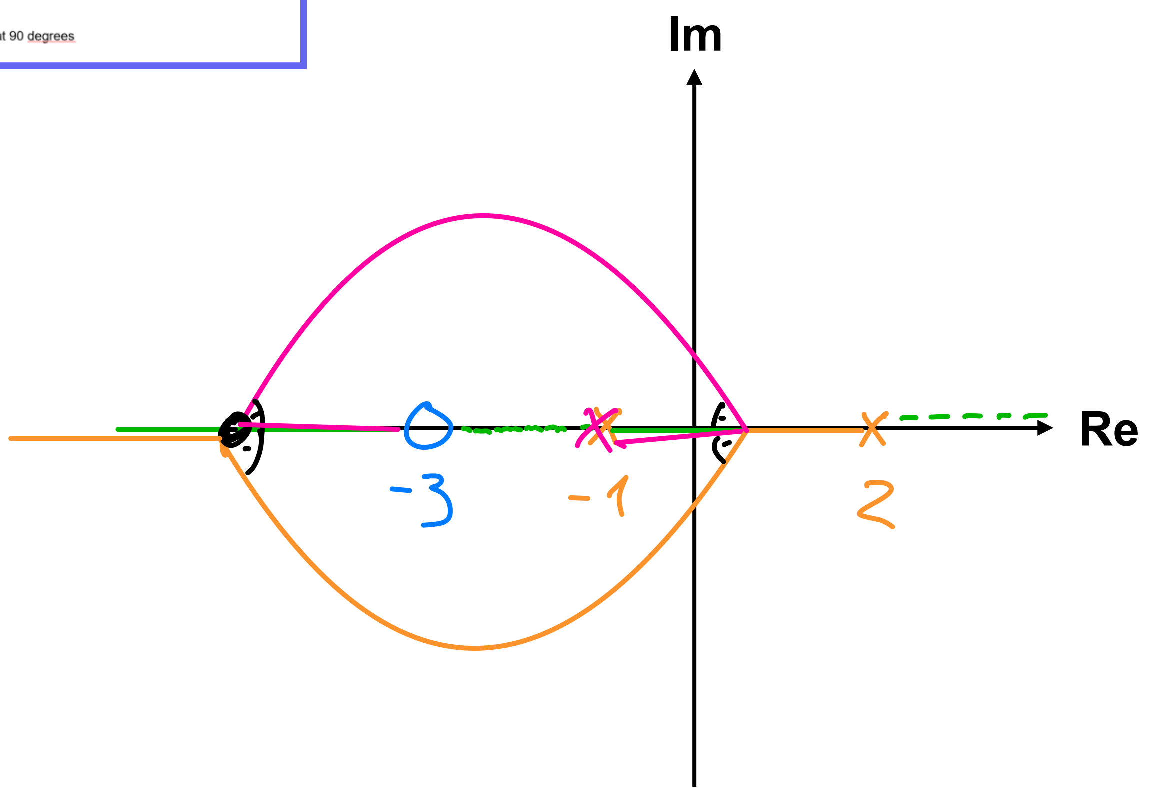
1. Lines enter and leave real axis at 90 degrees

* Poles: * Zeros:

* Poles: 2   * Zeros: 1

$$S_{com} = \frac{(2+(-4)) - (-1)}{1} = -1$$





$$\angle(s) = \frac{(2\overset{0}{q}+1)\cdot 180^\circ}{1}$$

$$= 180^\circ$$

Examples

Root Locus Rules

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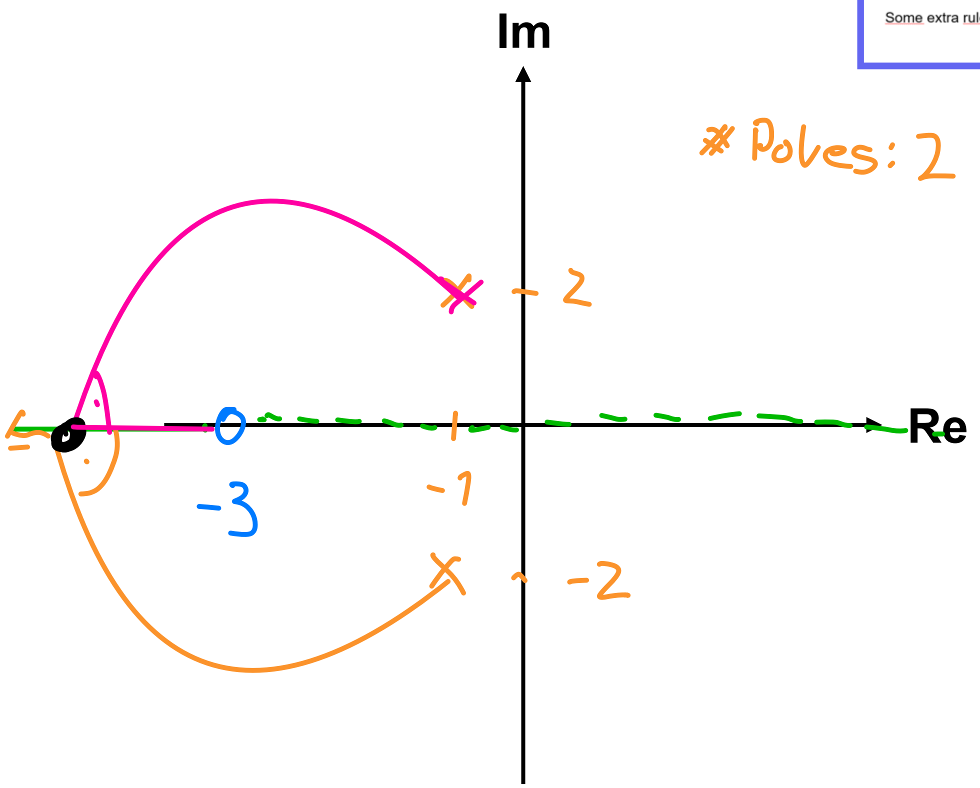
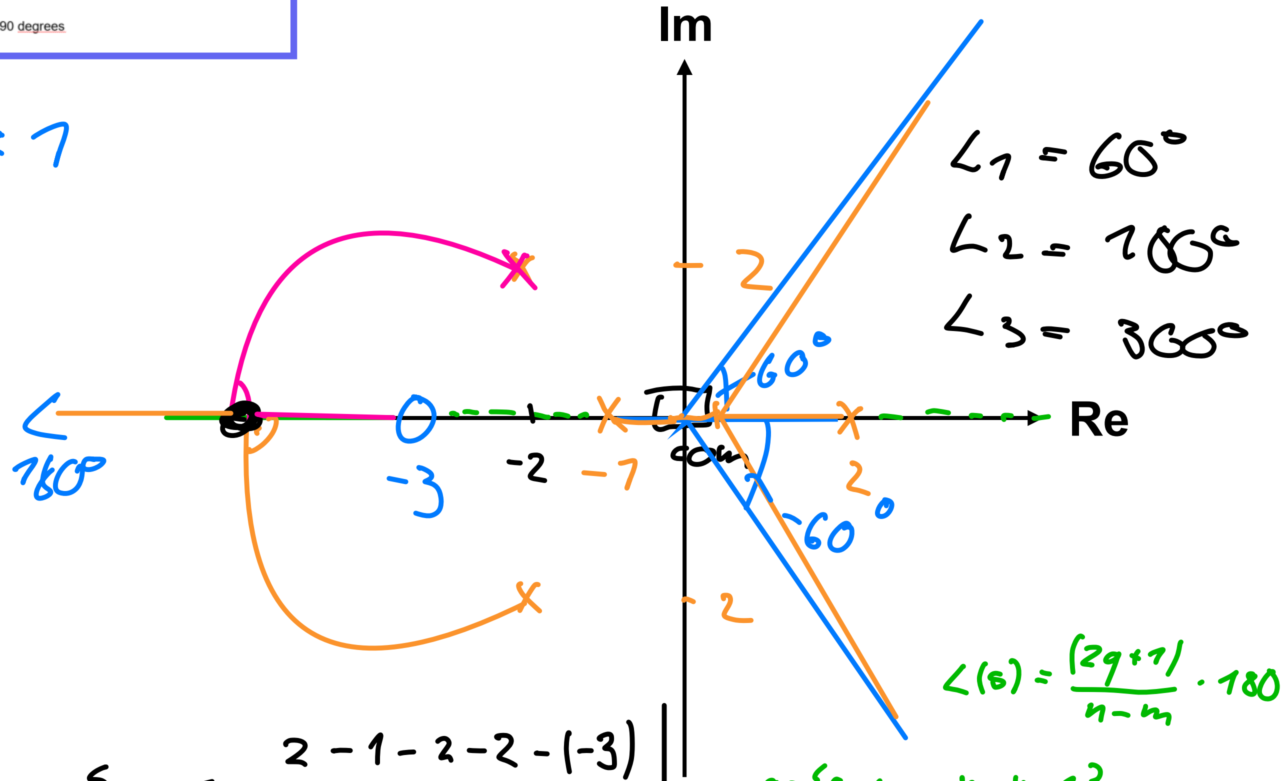




$$S_{com} = \frac{2 - 1 - 2 - 2 - (-3)}{3} = \frac{0}{3} = 0$$

$$\angle(s) = \frac{(2q+1)}{n-m} \cdot 180$$

$$q = \{0, 1, \dots \underbrace{n-m-1}_{2}\}$$

$$\Rightarrow \angle s_0 = \frac{2 \cdot 0 + 1}{3} \cdot 180 = 60$$

$$\angle s_1 = \frac{2 \cdot 1 + 1}{3} \cdot 180 = 180$$

$$\angle s_2 = \dots = 300$$

Some intution for RL Rules

Important Trick / Intuition

Don't get confused by the new notation of k being outside of L(s).
Next slide:

Generally we can write our systems O-L TF like this:

$$\boxed{kL(s) = k\frac{N(s)}{D(s)} = k\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}}$$

Then for the C-L TF:

$$\boxed{T(s) = \frac{kL(s)}{1+kL(s)} = \frac{kN(s)}{D(s)+kN(s)}}$$

And now finally, wanting to compute the poles, we have to solve

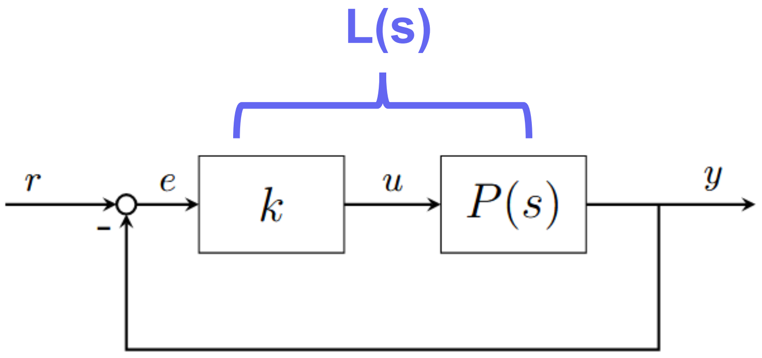
$$\boxed{D(s) + kN(s) = 0}$$

or equivalently

$$\boxed{\frac{N(s)}{D(s)} = -\frac{1}{k}}$$

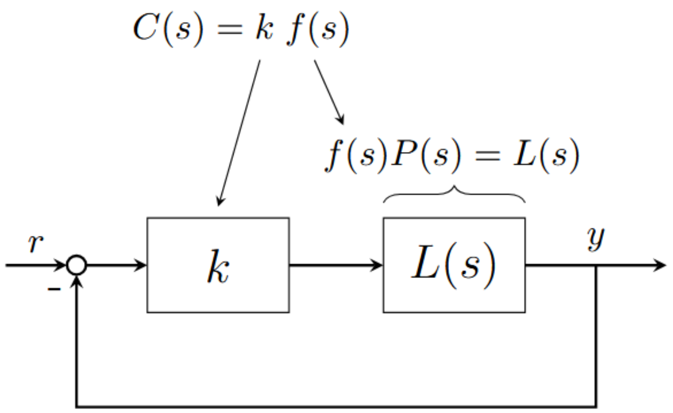
Notation difference

In the system we looked at, we set our Controller C(s) = k.
Therefore we set our O-L TF L(s) = kP(s)



or

Generally the Controller is of the form C(s) = k f(s). We can also just incorporate f(s) to our plant and call this then L(s). The k we will handle separately as a scalar gain. Just notation / interpretation, but **roughly the same**



Root Locus Rules

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Center of Asympt = $\boxed{S_{COM} = \frac{\sum x_p - \sum x_z}{n-m}}$

n = #Poles
m = #Zeros

$\boxed{\begin{array}{l} \angle(s) = \frac{(2q+1)\cdot 180^\circ}{n-m} \\ q = \{0, 1, ..., n-m-1\} \end{array}}$ = Angle of Asympt.

Some extra rule:

1. Lines enter and leave real axis at 90 degrees

1. and 2. Rule: Symmetry and Degree

$$\boxed{D(s) + kN(s) = 0}$$

Remember the equation above is used **to describe our closed loop poles.**

This can be rewritten as a general polynomial.

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0.$$

Therefore, either all roots are real, or complex roots come in complex conjugate pairs (ANA 1)
→ **symmetric w.r.t. real axis.**

Also, see how for the degree $\underbrace{\text{N} \leq \text{D}}_{\text{This was a restriction from LTI SISO Systems}}$, therefore the degree and number of poles stays the same

Root Locus Rules

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Center of Asympt = $\boxed{S_{COM} = \frac{\sum x_p - \sum x_z}{n-m}}$

n = #Poles
m = #Zeros

$$\boxed{\begin{gathered}\angle(s) = \frac{(2q+1)\cdot 180^\circ}{n-m} \\ q = \{0, 1, ..., n-m-1\}\end{gathered}}$$
= Angle of Asympt.

Some extra rule:

1. Lines enter and leave real axis at 90 degrees

3. Rule: Closed-Loop Poles

$$T(s) = \frac{kN(s)}{\boxed{D(s)+kN(s)}}$$

$$\boxed{D(s) + kN(s) = 0}$$

Remember that the equation above describes the poles of the closed-loop TF.
Lets now consider the two limit cases for k

For $k \rightarrow 0$ the closed-loop poles just **approach** the **open-loop poles**

$$\boxed{\lim_{k\rightarrow 0} D(s) + kN(s) = D(s)}$$

F $or\ k \rightarrow \infty$ the closed-loop poles **approach** the **open-loop zeros**

$$\boxed{\lim_{k\rightarrow \infty} \frac{1}{k} D(s) + N(s) \approx N(s)}$$

Root Locus Rules

We are only going to go one crucial step closer to the last 2 rules. To fully understand the derivation (not important), look in the lecture or read the script

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Center of Asympt = $\boxed{S_{COM} = \frac{\sum x_p - \sum x_z}{n-m}}$

n = #Poles
m = #Zeros

$$\boxed{\begin{array}{c}\angle(s) = \frac{(2q+1)\cdot 180^\circ}{n-m} \\ q = \{0, 1, ..., n-m-1\}\end{array}}$$ = Angle of Asympt.

Some extra rule:

1. Lines enter and leave real axis at 90 degrees

Magnitude and Angle Rule

$$T(s) = \frac{kN(s)}{\boxed{D(s)+kN(s)}}$$

$$\boxed{D(s) + kN(s) = 0}$$

To find the roots of our Closed Loop Transfer Function T(s), we want to solve the equation above.
We can also reformulate this into the equation below.

$$\boxed{\frac{N(s)}{D(s)} = -\frac{1}{k}}$$

As always, we can compare the two terms regarding their magnitude and their phase.

Magnitude and Angle Rule

Meaning: The Magnitude Rule states that for any point on the root locus the product of magnitudes of all zeros minus product of magnitudes of all poles must equate to the inverse absolute value of the proportional gain.

$$\left|\frac{N(s)}{D(s)}\right| = \left|-\frac{1}{k}\right| \quad \longrightarrow \quad \left|\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}\right| = \left|\frac{1}{k}\right|$$

Same for the phase / angle

$$\angle\frac{N(s)}{D(s)} = \angle\left(-\frac{1}{k}\right) \quad \longrightarrow \quad \begin{aligned}&\left[\angle(s-z_1)+\angle(s-z_2)+\cdots+\angle(s-z_m)\right]\\ &\quad -\left[\angle(s-p_1)+\angle(s-p_2)+\cdots+\angle(s-p_n)\right]\\ &= -(\angle 1 - \angle k).\end{aligned}$$

More Root Locus

Rules to follow (a little different than mine, but same)

Asymptotes

Contact point / Centroid of asymptotes

$$\boxed{S_{com} = \frac{\sum x_{Poles} - \sum x_{Zeros}}{\#Poles - \#Zeros}}$$

$$x_i \rightarrow Coordinates\ on\ the\ Real\ axis$$

Angle of asymptotes

$$\boxed{\alpha_n = \frac{2n+1}{\#Poles - \#Zeros} \cdot 180°}$$

$$n = \{0; 1; \dots; (\#Poles - \#Zeros - 1)\}$$

Sketching Rules

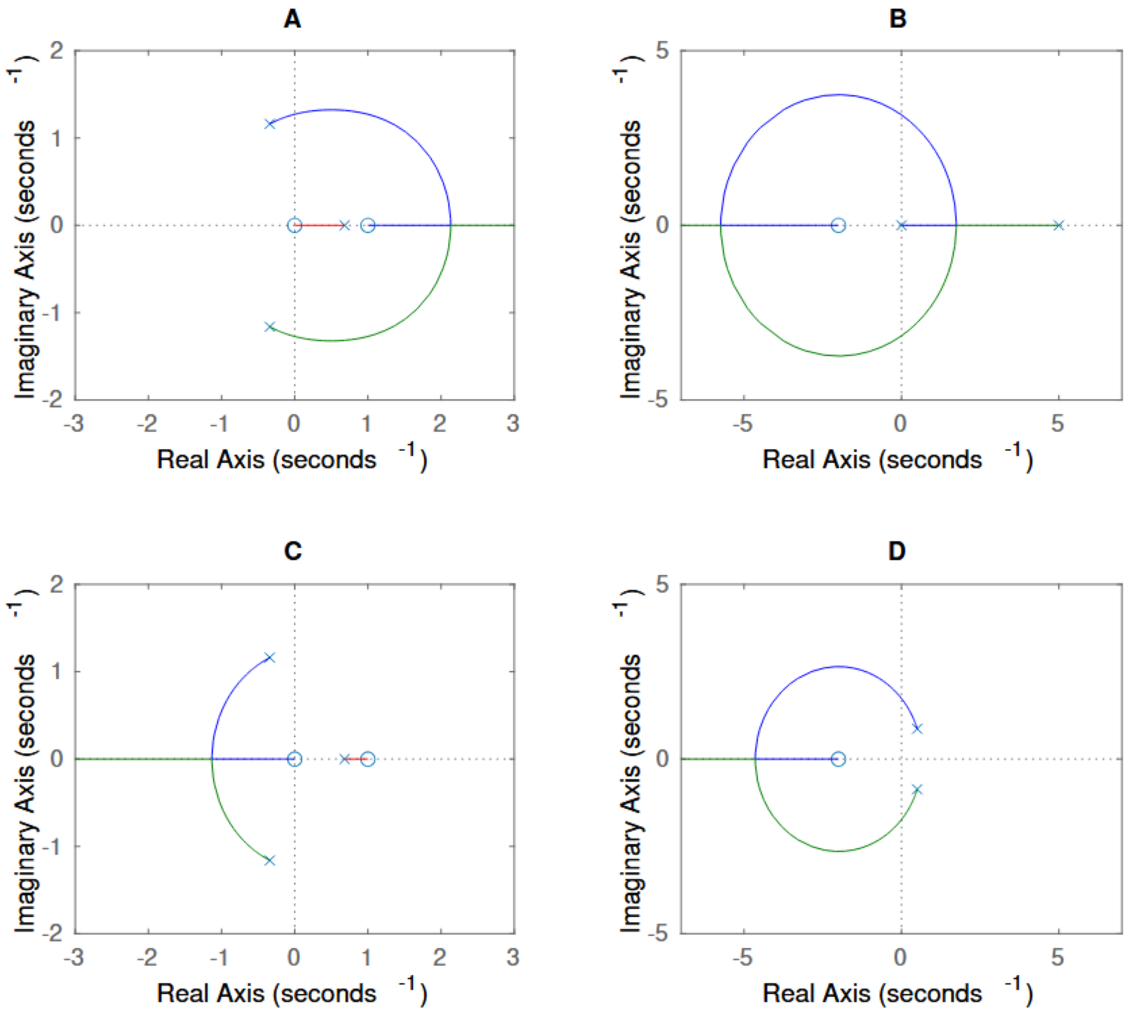
1. Root loci start at poles → go to zeros
2. There are n lines (loci) where n is the degree of Poles or Zeros (whichever is greater).
3. As k increases from 0 to ∞, the roots move from the poles of $G(s)$ to the zeros of $G(s)$.
4. When roots are complex, they occur in conjugate pairs.
5. At no time will the same root cross over its path.
6. The portion (Anteil) of the real axis to the left of an odd number of open loop poles and zeros are part of the loci. → **Roots are always sketched from the right to the left.**
7. Lines leave and enter the real axis at 90°.
8. If there are not enough poles or zeros to make a pair, then the extra lines go to / come from infinity.
9. Lines go to infinity along asymptotes.
10. If there are at least two lines to infinity, then the sum of all roots is constant.
11. K going from 0 to $-\infty$ can be drawn by reversing rule 5 and adding 180° to the asymptote angles.

Cool tool to sketch Root Locus https://lpsa.swarthmore.edu/Root_Locus/RLDraw.html

FS 2024

$$G_1(s) = \frac{s^2 - s}{s^3 + s - 1}, \; G_2(s) = \frac{s+2}{s^2 - s + 1}$$

$$G_3(s) = -\frac{s^2 - s}{s^3 + s - 1}, \; G_4(s) = \frac{s+2}{s(s-5)}$$



A) (A, G_3), (B, G_4), (C, G_2), (D, G_1)

B) (A, G_3), (B, G_2), (C, G_1), (D, G_4)

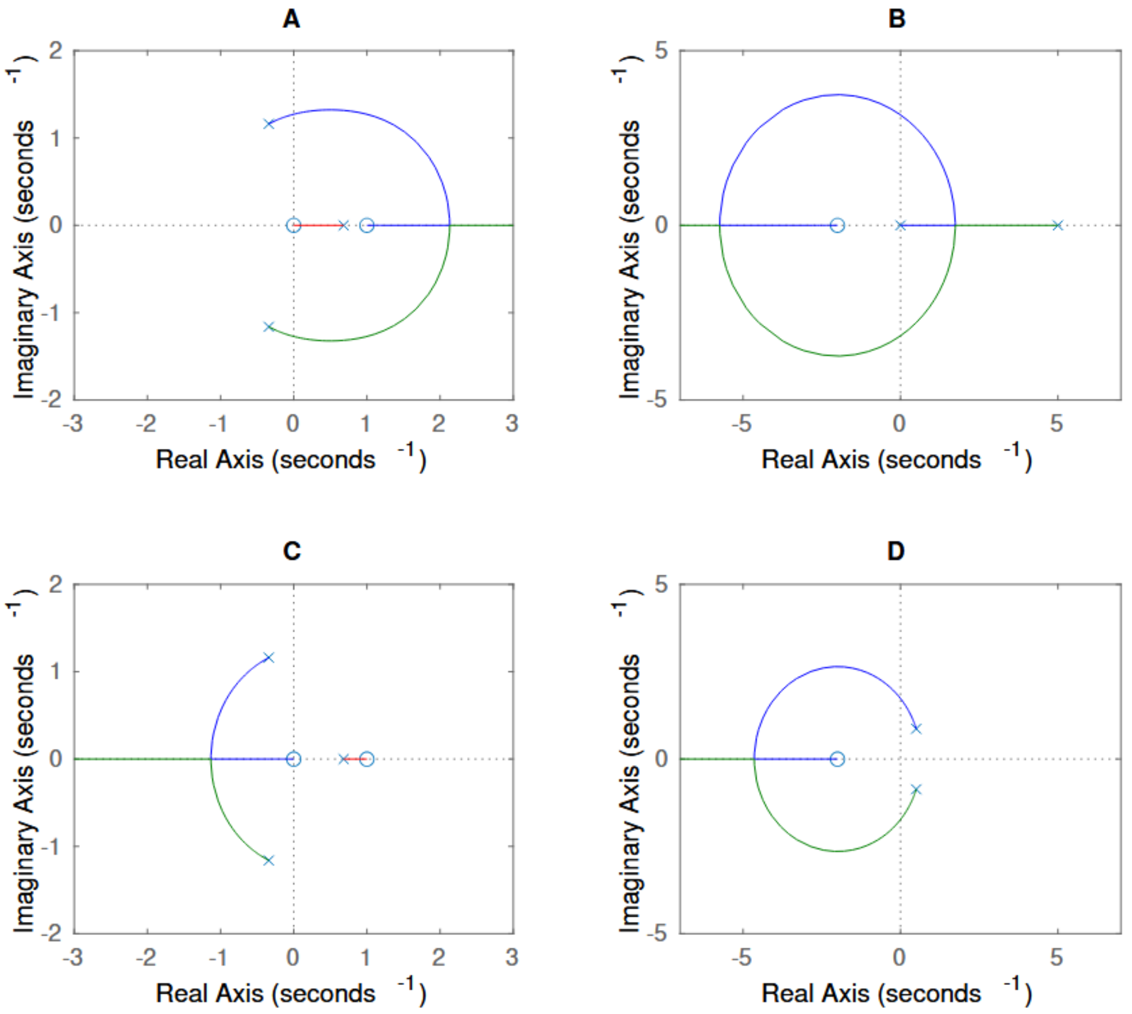
C) (A, G_1), (B, G_2), (C, G_3), (D, G_4)

D) (A, G_3), (B, G_4), (C, G_1), (D, G_2)

FS 2024

$$G_1(s) = \frac{s^2 - s}{s^3 + s - 1}, \quad G_2(s) = \frac{s + 2}{s^2 - s + 1}$$

$$G_3(s) = -\frac{s^2 - s}{s^3 + s - 1}, \quad G_4(s) = \frac{s + 2}{s(s - 5)}$$



A) (A, G_3), (B, G_4), (C, G_2), (D, G_1)

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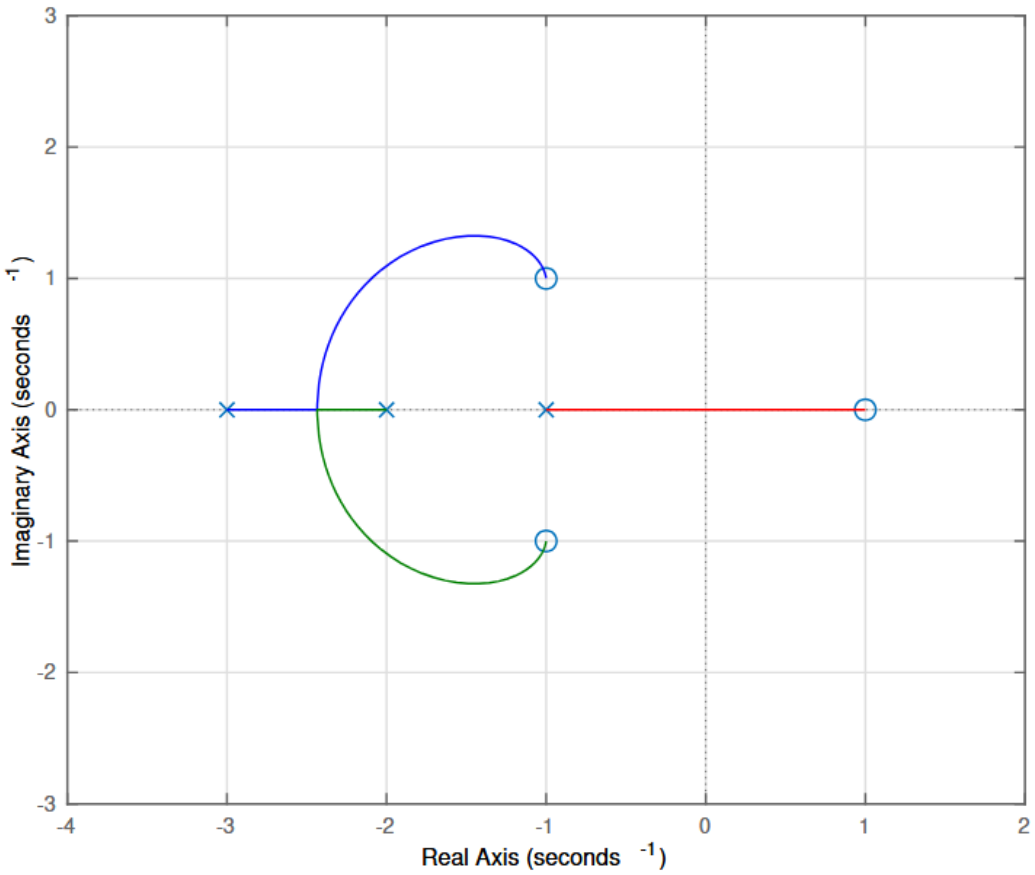
C) (A, G_1), (B, G_2), (C, G_3), (D, G_4)

D) (A, G_3), (B, G_4), (C, G_1), (D, G_2)

FS 2024

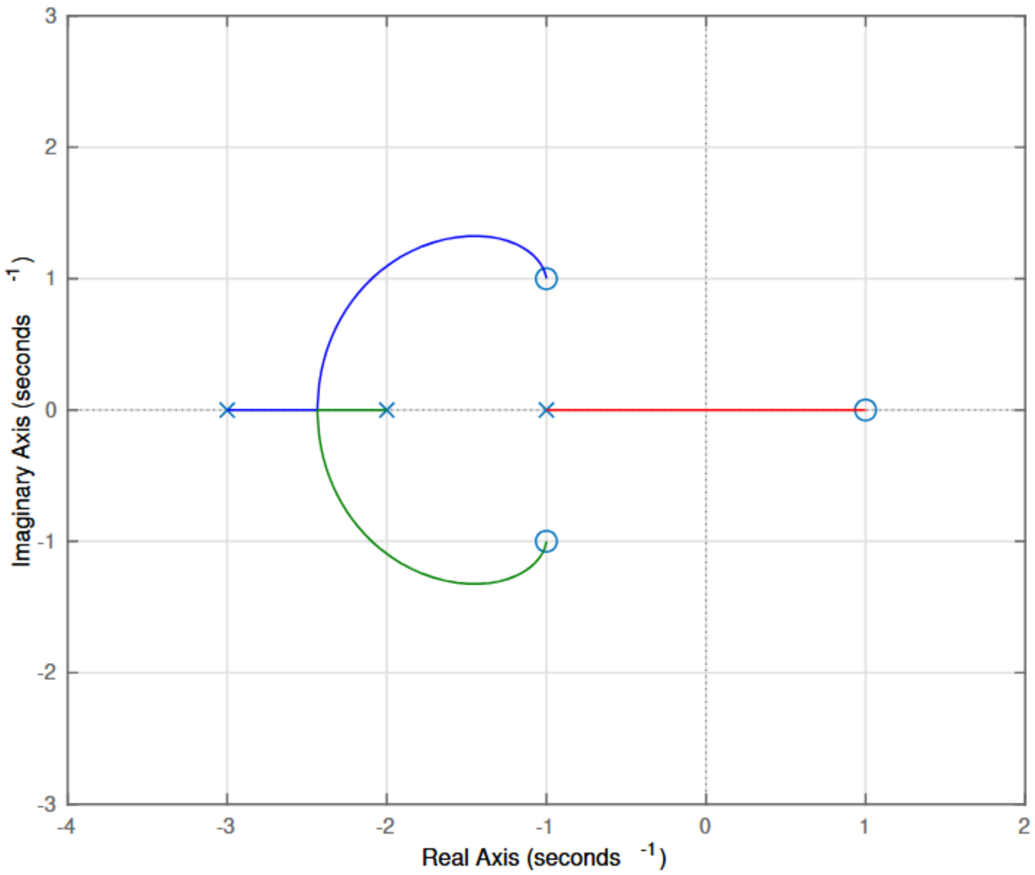


Mark the correct statements

1. The open-loop system L is a minimum-phase system.
2. The open-loop system L is asymptotically stable.
3. There exists a gain k^* such that for all k such that $0 < k \leq k^*$ the closed-loop system T is asymptotically stable and all poles of the closed-loop system have zero imaginary part, i.e. are real numbers.

1 2 3

FS 2024



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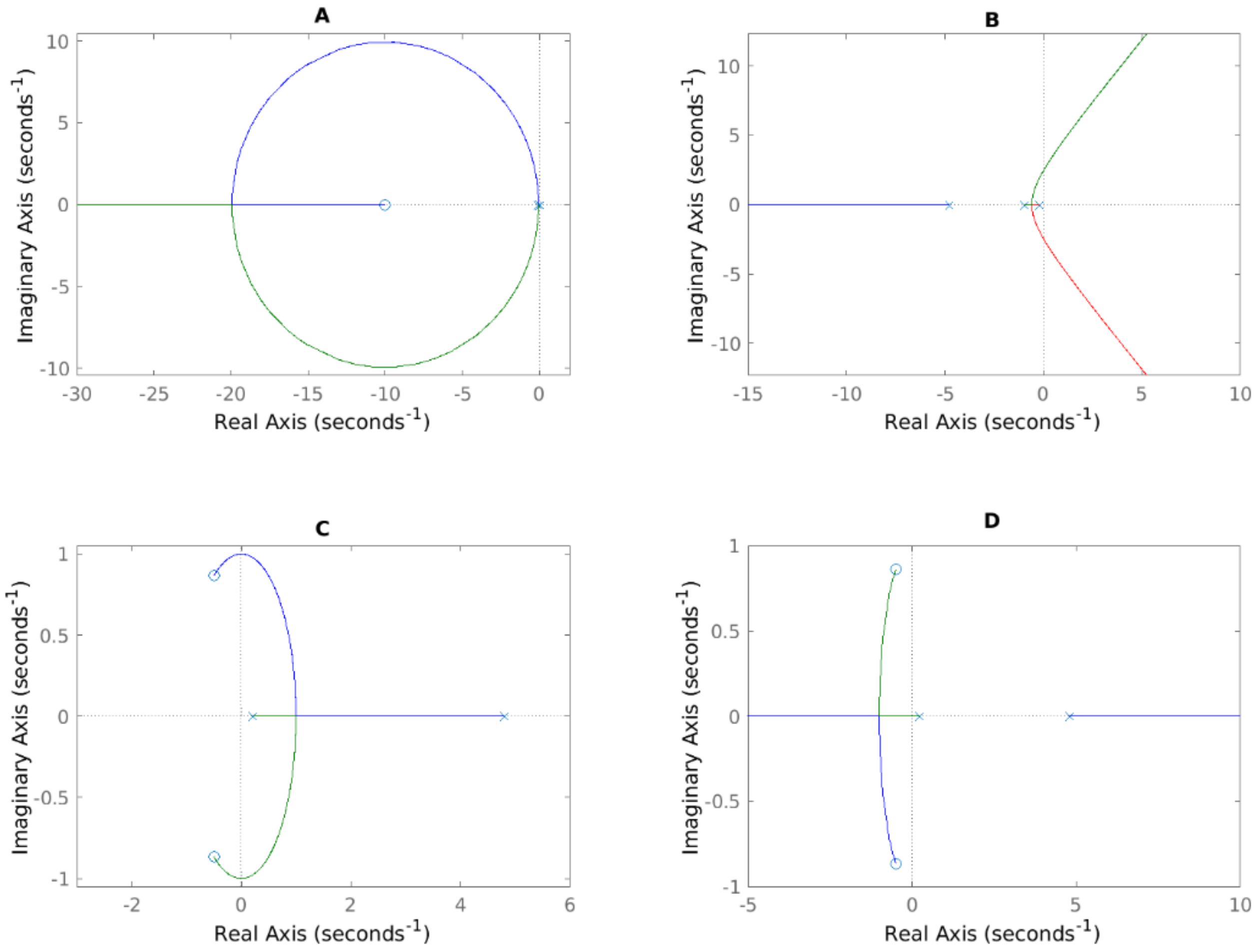
1 2 3

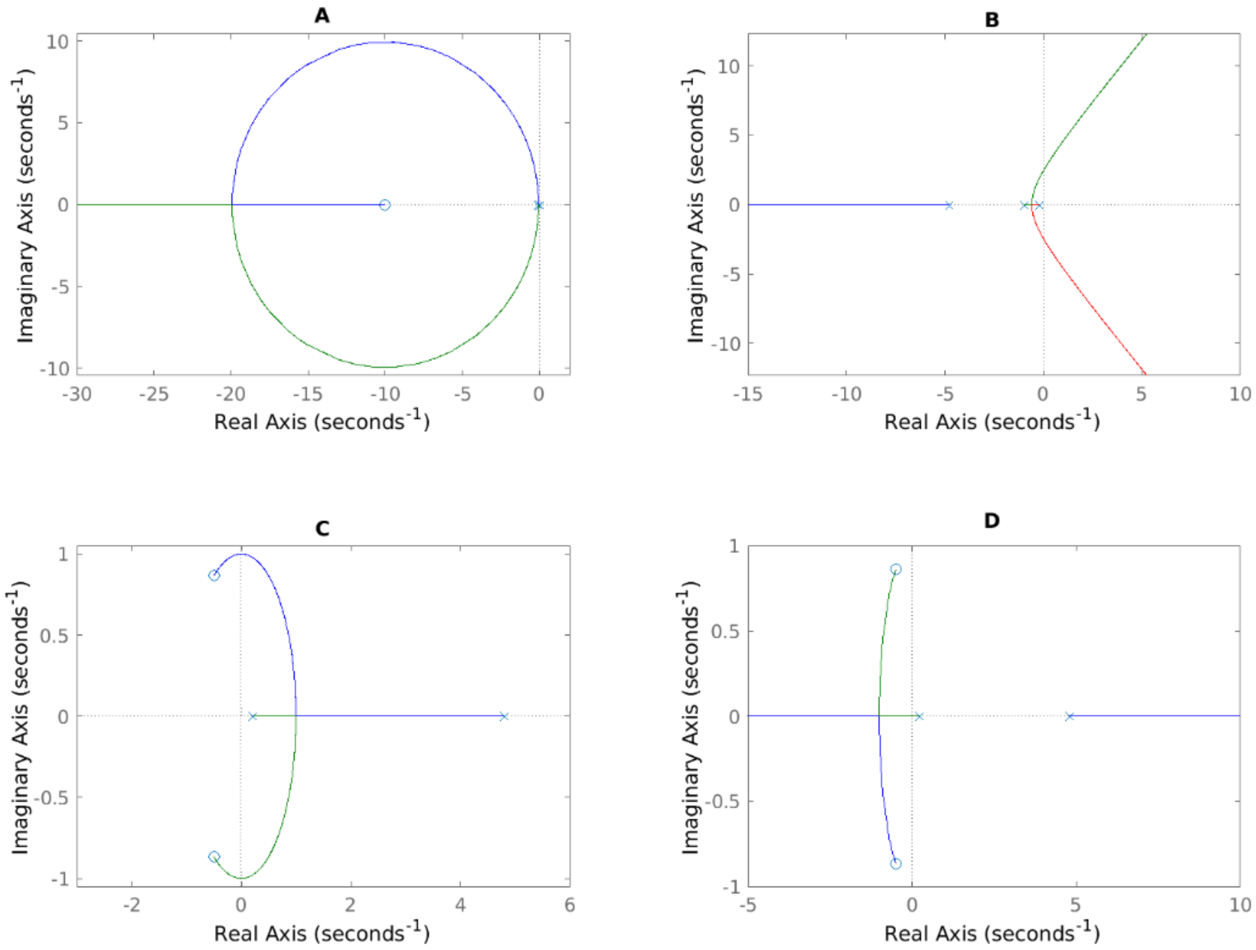
HS 2022



(1.5 Points) Assign each transfer function to the corresponding root locus plot.

Transfer Function	A	B	C	D
$L_1(s) = \frac{1}{(s+1)(s^2+5s+1)}$				
$L_2(s) = -\frac{s^2+s+1}{s^2-5s+1}$				
$L_3(s) = \frac{s+10}{(s+0.01)(s+0.1)}$				
$L_4(s) = \frac{s^2+s+1}{s^2-5s+1}$				

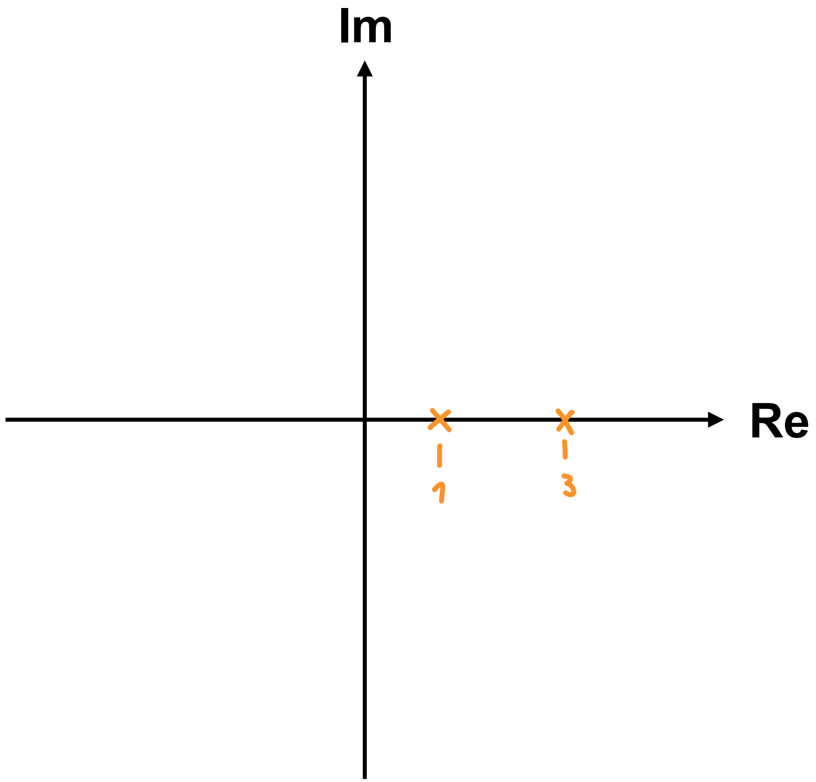
HS 2022



Transfer Function	A	B	C	D
$L_1(s) = \dfrac{1}{(s+1)(s^2+5s+1)}$		x		
$L_2(s) = -\dfrac{s^2+s+1}{s^2-5s+1}$				x
$L_3(s) = \dfrac{s+10}{(s+0.01)(s+0.1)}$	x			
$L_4(s) = \dfrac{s^2+s+1}{s^2-5s+1}$			x	

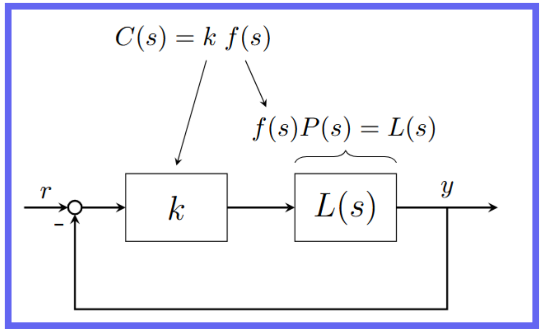
More Complex Controller

Motivation Dynamic Compensators

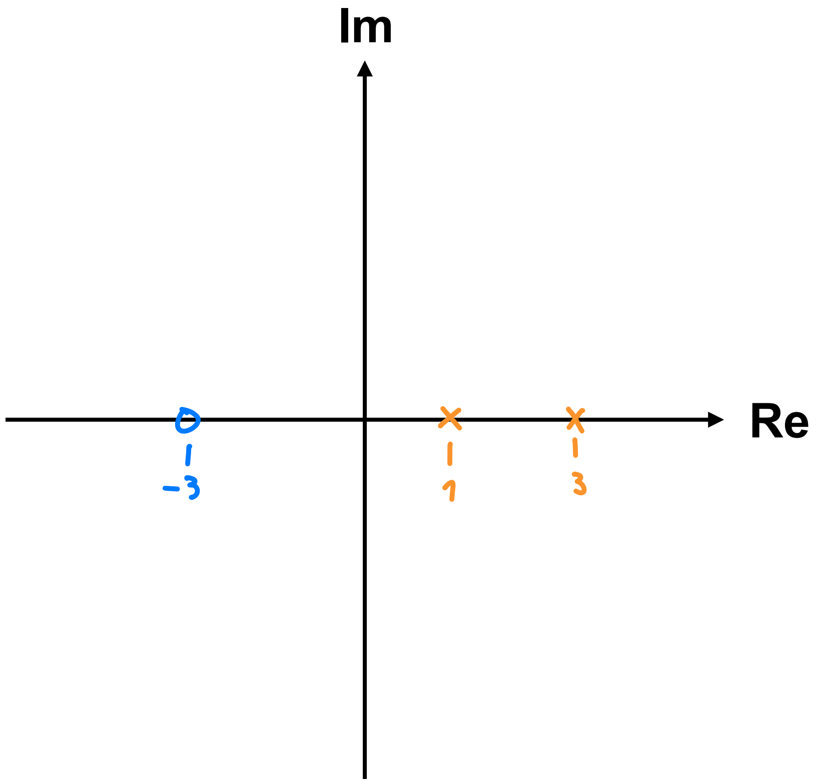


We can see that for no value of k, the poles become stable. So what can we do?

We know that by designing our controller differently, we can influence the TF in different ways.



Motivation Dynamic Compensators



So what happens when we place a zero to the left?

Q&A Session / Done

Feedback

jschultev.github.io/personal_website/Feedback